# Two-dimensional sandpile model with stochastic slide

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A theoretical two-dimensional sandpile model with stochastic slide is proposed. Numerical simulations show that the behavior of this model is different from that of previous models. Specifically, there exists a damping length scale, which is in agreement with a real sandpile experiment. The parameter dependence of the damping length scale is also discussed.

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## I. INTRODUCTION

Recently Bak, Tang, and Wiesenfeld introduced a concept of self-organized criticality [1] in the investigation of extended dissipative dynamical systems. They showed that such dynamical systems naturally evolve into a critical state through a self-organization process. This state is barely stable and, when it is perturbed, the resulting relaxation processes are scale invariant. They suggested that long-range temporal correlation with a $1/f$ power spectrum could be understood in terms of self-organized criticality and that there is connection between $1/f$ noise and the spatial self-similar structure of the critical state [1]. Since then, many physicists have studied various kinds of such systems. Especially the theoretical sandpiles are studied extensively as a paradigm for self-organized criticality [1–4].

Several cellular-automaton models of sandpiles have been shown both numerically and analytically [4] to exhibit scale invariance under generic conditions. And all these models have a common feature: the slide is deterministic though the particle may be added at a random site. In these models, when the sandpile reaches the critical state, the scaling behavior and universality have been discussed. Assuming $s$ denotes the toppling size of the sandpile, and $\delta$ the number of grains that drop off the edge when a particle is added, one can find in a large class of sandpiles that the distribution functions $D(s)$ and $F(\delta)$ obey the following scaling laws:

$$D(s) \propto s^{-\alpha} \tag{1}$$

and

$$F(\delta) \propto \delta^{-\beta} , \tag{2}$$

with $s$ or $\delta$ varying in a quite wide range. Here $\alpha$ and $\beta$ are constants, and both $D(s)$ and $F(\delta)$ are normalized to the total number of events of adding particles. Since the simulations are always on systems of finite size, it is important to consider the finite-size effect. With these taken into account, these two distribution functions are fitted by the finite-size scaling method or multifractal-analysis method by some authors [1,2].

However, experiments [5–7] which examined the existence of the critical state in sandpiles draw different conclusions. An experiment [5] showed that the avalanches occurred with a mean size and lifetime, which did not show any critical behavior at the angle of repose, in contrast to the theoretical models. A recent experiment [6], however, demonstrated the existence of a critical state in the evolution of a sandpile. In a similar incline ramping experiment [7], a power-law distribution of size for smaller avalanches occurred, and the number of these small avalanches followed a power law, which demonstrated that the system was of critical behavior. However, in the experiment of Ref. [6], the sandpile was built up to a critical size and then perturbed by the addition of individual grains of sand onto the pile. After each grain was added, the size of the resulting avalanche, if any, was obtained by recording the total mass of the pile. For small sandpiles, the scaling law (2) holds, and the distribution functions $F(\delta,L)$ of different linear size $L$ of piles exhibit finite-size scaling. This suggests that these sandpiles are in a self-organized critical state. For larger sandpiles, however, the scaling law (2) breaks down, and the distribution function $F(\delta)$ becomes sharply peaked at nonzero values of $\delta$. Therefore, the criticality breaks down in the limit of large sandpiles, in contrast to the theoretical models in Refs. [1–4]. According to their experiment, the authors of Ref. [6] believed that the presence or absence of self-organized criticality was related to a damping length scale which had not been fully understood. As far as we know, the power law (1) has not been checked experimentally, and the damping length scale has not been found in theoretical models yet.

When we talk about the deterministic nature of the slide, it means that as long as the slope (or height) at every site of a sandpile is known, the avalanche will lead the sandpile to unique stable state. However, in a real sandpile, the behavior is different [8]. Since the grains may have different shapes and sizes, and the direction and magnitude of the force acting on grains of a given site might be different, the dynamics of an evolving sandpile could be very complicated. So it will be reasonable to introduce a stochastic nature in the evolution processes of a sandpile. On the basis of this, we propose an alternative theoretical model. The main results of this model have already appeared in a previous Rapid Communication [9].

The present model differs in two ways from the previous ones. First, the region over which the grains are added is confined to a certain part of the sandpile, instead of adding randomly on the whole pile; this is in agreement with the experimental arrangement [6], where the grains are dropped onto the top of a sandpile. Second, the slide is stochastic instead of deterministic. To characterize this feature, the threshold value at a site is allowed to change when this site accepts grains sliding from its neighbors. This difference makes the behavior of the model quite different from that of the model with deterministic slide [1,2]. Comparing the results of this model with those of the experiment [6], one may find that our model is much closer to the real sandpile. By investigating this model we could understand more about the evolution processes in real sandpiles.

## II. THE MODEL

In the present model, the sandpile is built up on a base of an isosceles triangle. The basic variable $h(x,y)$ in the model is the height of the sandpile at the site $(x,y)$, here $x \geq 0$, $y \geq 0$, and $x+y<L$. Boundary conditions are such that the grains can flow out of the system at the bottom side $x+y=L$ only. We define a vector field $(Z_x, Z_y)$ with

$$Z_x = h(x,y) - h(x+1,y) \; , \qquad Z_y = h(x,y) - h(x,y+1) \; , \tag{3}$$

and denote $Z = \max(Z_x, Z_y)$ as the slope. We define also a threshold $Z_c$ which determines whether a site is stable. In order to make our model closer to the experiment [6], we assume that the added grains are confined to the top part of the sandpile $x+y<cL$, with $0 \leq c < 1$. When a grain is added at a certain site, the height of this site will increase by a unit. So the addition of a grain at site $(x,y)$ is expressed by

$$h(x,y) \rightarrow h(x,y)+1 \quad \text{for } x+y<cL \; . \tag{4}$$

If the slope at some site $(x,y)$ exceeds the threshold $Z_c$,

$$Z > Z_c \; , \tag{5}$$

this site becomes unstable and will give $s = Z - Z_c$ grains to the lower one of its two nearest-neighbor sites at $(x+1,y)$ or $(x,y+1)$, then

$$h(x,y) \rightarrow h(x,y) - s \tag{6}$$

and

$$h(x+1,y) \rightarrow h(x+1,y) + s \quad \text{for } h(x+1,y) > h(x,y+1) \; , \tag{7}$$

$$h(x,y+1) \rightarrow h(x,y+1) + s \quad \text{for } h(x,y+1) > h(x+1,y) \; .$$

If the two neighbors have the same height, the grains topple to either site with equal probability. The open boundary conditions are

$$h(x,y) = 0 \quad \text{for } x+y=L \; . \tag{8}$$

In order to introduce a stochastic slide in the evolution processes of the sandpile, we assume that the threshold $Z_c$ at site $(x,y)$ could change according to the number of grains falling from its neighbors, $(x-1,y)$ and $(x,y-1)$. In other words, when some particles slide from its neighbors of a given site, this site must obtain a certain quantity of momentum. So the particles at this site might be easier to topple. Therefore the threshold at this site may become lower. Let $p_s(r)$ be the probability that the threshold $Z_c$ becomes $Z_c - r$ when the number of particles sliding from its neighbors is $s$. Certainly, there is considerable freedom in the choice of a form for the function $p_s(r)$. In general, we know that $p_s(r) \geq 0$, $p_0(r) = \delta_{r0}$. Since the more grains fall the lower the threshold will be, the average of $r$, i.e., $\langle r \rangle = \sum_{r=1}^{\infty} r p_s(r)$, must be a monotonically increasing function of $s$. Moreover, $\langle r \rangle$ must be bounded. In this paper we first choose the following simple form for the function $p_s(r)$ for $s>0$:

$$p_s(r) = \begin{cases} 0 & \text{for } r<0 \text{ or } r>r_c \\ 1-\lambda & \text{for } r=0 \\ \lambda/s & \text{for } 1 \leq r \leq r_c - 1 \\ (s - r_c + 1)\lambda/s & \text{for } r = r_c < s+1 \; , \end{cases} \tag{9}$$

with two parameters $\lambda$ and $r_c$. The values of the two parameters could vary in $0 \leq \lambda \leq 1$ and $0 \leq r_c \leq Z_c$. When $\lambda \rightarrow 0$, this model comes back to a trivial two-dimensional model [2], which has a minimally stable state, and has no self-organized criticality.

In the following sections we will discuss the influence of the form of the function $p_s(r)$ upon the properties of sandpiles.

## III. THE DAMPING LENGTH SCALE

In Ref. [9], many results, which conform to those of the experiment in Ref. [6], were drawn in the case of the above model with $c=0.5$, $\lambda=0.2$, $r_c=5$. One of the most important results is that there exists a damping length scale, which was considered to relate to the presence or absence of self-organized criticality in a real sandpile by the authors of Ref. [6].

As shown in Ref. [9], $\langle \Delta T_\delta \rangle$ increases with the base size $L$ of the sandpile; here $\langle \Delta T_\delta \rangle$ is the average time interval between the successive events of the mass flow off the sandpile. The value of the exponent $\gamma$ $(\gamma = d \ln \langle \Delta T_\delta \rangle / d \ln L)$ is not always the same for different base sizes $L$ of the sandpile. There exists a turning point in the curve of $\ln\langle T_\delta \rangle$ versus $\ln L$ (see Fig. 2 in Ref. [9]). This turning point corresponds to a base size $L_c$ of the sandpile which is called the damping length scale. The value of $\gamma$ is very large when $L < L_c$, but much smaller when $L > L_c$. However, there is no damping length scale in the BTV model of two dimensions in Ref. [1]. In the next section we will see that the existence of the damping length scale is due to the stochastic nature of the slide.

The large value of $\gamma$ when $L < L_c$ could explain why the behaviors of the fluctuations in total mass $M$ of sandpiles are quite different for the sandpiles with different base sizes (see Fig. 4 in Ref. [9]). Because the value of the exponent $\gamma$ is very large when $L < L_c$, the average time interval $\langle \Delta T_\delta \rangle$ will increase rapidly when the size of the sandpile increases. So the number of avalanches occurring will become smaller while the average size of avalanches will become larger in the same time interval, for when the total number of adding grains are the same. This is in agreement with the experiment in Ref. [6]. Therefore, one has to measure for a longer time to get sufficient information about the evolution of $M$ for sandpiles with large bases.

In addition, because of the existence of the damping length scale, and the value of $\gamma$ considerably large at $L < L_c$ while small at $L > L_c$, the scaling behavior of the distribution functions $F(\delta, L)$ of different base sizes $L$ of sandpiles are different from the theoretical models discussed previously [2], here $\delta$, as mentioned above, is the total number of grains that flow off the system when a grain is added. Multifractal analysis of $F(\delta, L)$ shows that the sandpiles with different sizes can be divided into two groups, see Fig. 3(b) in Ref. [9]. The first group comprises sandpiles of small size, $L < L_c$. The second group consists of the sandpiles with $L > L_c$. Therefore, we believe that the sandpiles with base sizes smaller or larger than $L_c$ belong to different classes of universality.

Like the results measured in the experiment [6], the mass power spectrum of the time series have the trivial form $1/f^2$ for the present choice of parameters. This is in agreement with the theoretical analysis in Ref. [10], and consistent with the power spectrum of a weighted random walk.

## IV. THE EXPLANATION FOR THE EXISTENCE OF THE DAMPING LENGTH SCALE

The reason for the existence of the damping length scale is nothing but the stochastic nature in the sliding process. At the beginning of an avalanche caused by adding a single grain, the threshold must be $Z_c$. With every step in the toppling process, there is some chance to lower the threshold. The larger the size $L$ of the sandpile there is, the more events of decreasing the threshold will occur, and the lower threshold $Z_c - r$ will be generated. This will certainly make the size of the avalanche larger, and cause a larger time interval $\langle \Delta T_\delta \rangle$. Hence the value of $\gamma$ could be larger for $L < L_c$. Let us use $P_c(x,y)$ to represent the average probability density with which the threshold at site $(x,y)$ becomes $Z_c - r_c$, the lowest one,

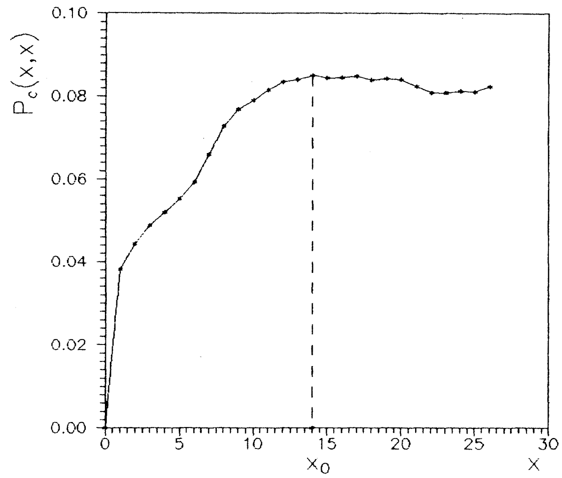


FIG. 1. The probability density $P_c(x,x)$ will reach saturation as $x$ increases. Here $c=0.2$, $\lambda=0.2$, $r_c=5$, the saturation point site $x_0$ is about 14, and the corresponding damping length scale $L_c$ is about 28.

when this site has accepted grains sliding from its neighbors in the sliding processes for a given sandpile with base size $L$. We will see that $P_c(x,y)$ may reach saturation as grains slide from the top to the open boundary.

Note that $P_c(x,y)$ cannot exceed the value of $\lambda$ in Eq. (9). To see the saturation property clearly, we plot $P_c(x,y)$ versus $(x,y)$ along the line $x=y$ in Fig. 1. This curve can represent the change of $P_c(x,y)$ typically. $P_c(x,x)$ will reach a saturation value at a certain site $x_0$, corresponding roughly to the damping length scale $L_c$. At the upstream of this site $x < x_0$, $P_c(x,x)$ increases with $x$. In other words, the probability with the lowest threshold increases in the sliding process, the total number of sites in the whole sandpile where their threshold is lowest becomes more, and these sites topple more easily. So the average avalanche size could increase rapidly with $L$, and the average time interval $\langle \Delta T_\delta \rangle$ could increase rapidly with $L$. Hence the exponent $\gamma$ could keep large at $L < L_c$. However, at the downstream of this site $x > x_0$, $P_c(x,x)$ remains almost unchanged. That is to say, the probability density for a site with the lowest threshold does not increase in the sliding process, and the total number of the sites in the whole sandpile where threshold is the lowest has reached saturation. So the avalanche size and therefore the average interval time $\langle \Delta T_\delta \rangle$ do not increase rapidly with $L$ when $L > L_c$. Hence the slope $\gamma$ must be smaller for the case of $L > L_c$. Because the slide in a real sandpile would be of a stochastic nature, we expect that the qualitative characteristic of the curve in Fig. 2 of Ref. [9] could be verified experimentally. Based on the experiment of Ref. [6], we believe that the relation between $\langle \Delta T_\delta \rangle$ and $L$ predicted in this model is qualitatively correct.

## V. THE PARAMETER DEPENDENCE OF THE DAMPING LENGTH SCALE

Because of the introduction of stochastic nature in the sliding process, there exists a damping length scale $L_c$.

TABLE I. The damping length scale $L_c$ decreases as $\lambda$ increases with the value of $r_c$ and $c$ fixed.

| $\lambda$ | 0.1 | 0.2 | 0.4 | 0.6 | 0.8 |
|---|---|---|---|---|---|
| $L_c$ | 36 | 32 | 24 | 20 | 18 |

The existence of $L_c$ is the important characteristic of the present model that is different from previous models. In the above section, the numerical simulation results were shown for the value of parameters $c=0.5$, $\lambda=0.2$, $r_c=5$ for model (9). Now we pay attention to the parameter dependence of the damping length scale.

From the forms of model (9), we can see that the larger the value of $\lambda$ is, the easier the slide occurs. In other words, the larger the value of $\lambda$ is, the faster the $P_c(x,x)$ reaches its saturation value. Therefore, the corresponding value of the damping length scale $L_c$ becomes small. In Table I we give the values of the damping length scale $L_c$ for different values of the parameter $\lambda$ when the values of $r_c$ and $c$ are fixed. From this table, one can see that the damping length scale $L_c$ decreases as $\lambda$ increases.

The change of the value of $c$ also influences the damping length scale. The results in Fig. 2 show that the damping length scale $L_c$ becomes larger when the value of $c$ changes from 0.2 to 0.5. The increase of the value $c$ means that the region over which grains are added becomes larger, the grains could be added closer to the open edge of the sandpile, and the events of lowering the threshold led by these grains would have smaller probability. $P_c(x,x)$ will reach a saturation value more slowly when $c$ increases. So the corresponding damping length scale $L_c$ increases.

Finally, the avalanche size $\delta$ is also closely related to the value of $r_c$. Results are shown in Fig. 3. When the value of $r_c$ increases, the threshold can be much lower, and the grains topple more easily. So the avalanche size can become much larger, and therefore the average interval time $\langle \Delta T_\delta \rangle$ will increase with $L$ faster, then the slope of the curve $\langle \Delta T_\delta \rangle$ versus $L$ for $L < L_c$ becomes larger when $r_c$ increases.

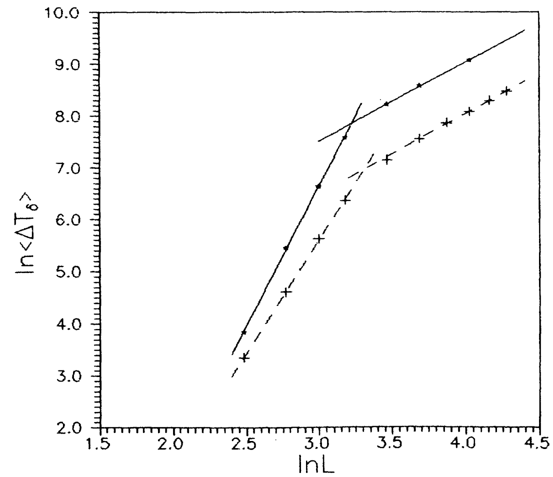


FIG. 3. The value of the slope for the part $L < L_c$ increases when the value $r_c$ increases from 5 to 10. The curves for $r_c=5$ are shown by dashed lines and for $r_c=10$ by solid lines. Here $\lambda=0.2$ and $c=0.2$.

In general, the damping length scale is closely related to the value of these three parameters.

## VI. DISCUSSION AND CONCLUSION

The stochastic nature of the sliding process is the reason for the existence of the damping length scale. So different forms of the function $p_s(r)$ should not change the properties of the model qualitatively. To prove this, in numerical simulations we select another function for $p_s(r)$:

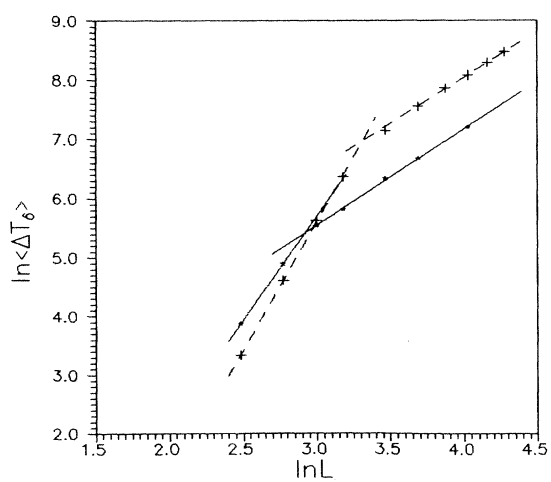


FIG. 2. The damping length scale $L_c$ decreases as the value $c$ decreases from 0.5 to 0.2. The dashed line is for $c=0.5$. The solid line is for $c=0.2$. Here $\lambda=0.2$, $r_c=5$.

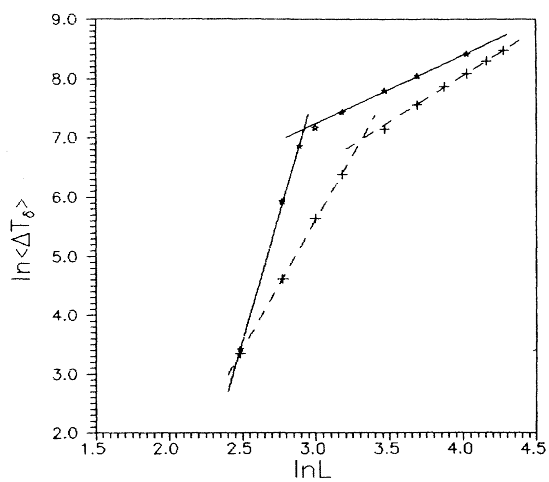


FIG. 4. The value of slope for the part $L < L_c$ in model (13) is larger than that in model (9). The solid line indicates model (13).

$$p_s(r)=\begin{cases}(1-u)u & \text{for } r<r_c\\ u^{r_c} & \text{for } r=r_c\\ 0 & \text{for } r>r_c\ ,\end{cases} \tag{10}$$

where $u=\exp(-1/s\lambda)$. Figure 4 shows the relations between the average interval $\langle \Delta T_\delta \rangle$ and the base size of the sandpile $L$ for this model. For comparison, the curve $\langle \Delta T_\delta \rangle$ versus $L$ for model (9) with the same value of parameters is also given. One can see that the damping length scale still exists for model (10), but the value of damping length scale is different from the previous one, and the value of the slope in the part of $L<L_c$ for model (10) is apparently larger than that for model (9). The numerical simulation also shows that other properties of this model are qualitatively the same as model (9).

In conclusion, we introduced an alternative kind of theoretical sandpile model. The main difference from previous models is that the slide is stochastic instead of deterministic. This difference makes the behavior of the model here change in many ways, especially, the existence of the damping length scale in our model. It does not exist in any one of the previous models, but is in agreement with the real sandpile experiment [6]. Besides this, the universality and scaling properties of the model here are also different from those of previous models. Because the exponent $\gamma$ is considerably large at the part of $L<L_c$, the average interval time $\langle \Delta T_\delta \rangle$ increases rapidly with the size of the sandpile, and a more time-consuming experiment is expected for verifying the critical behavior for larger sizes of sandpiles.

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